Applied economic networks - VI

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October 25, 2021

Today's class

- Network games - continuation
- Amine's presentation: *Peer effects and social networks in education*

A more interesting game

- Each player $i \in N = \{1, 2, \ldots, n\}$ simultaneously selects an action $a_i \geq 0$.
- Payoff depend on everyone's action $u_i(a_1, a_2, \ldots, a_n)$.

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Best response

$$BR_i(\mathbf{a}_i) = \alpha \sum_j w_{ij} a_j + b_i$$

where $\alpha > 0$.

What if $w_{ij} = \frac{1}{d_i}$ for all j such that $g_{ij} = 1$?

Why this best response?

- Can be shown that it corresponds to the linear-in-means model (peer effects)
- In a dynamic setting this is the DeGroot updating rule (naive learning – in two lessons)
- In general has deep interpretation
- It is linear..

What payoff?

The best response described before can be derived from two different payoff functions:

- $$u_i(\mathbf{a}) = -\frac{a_i^2}{2} + \alpha \sum_j w_{ij} a_i a_j + b_i a_i$$
- By setting $b_i = (1-\alpha)y_i$:

$$u_i(\mathbf{a}) = -\alpha \sum_j (a_i - a_j)^2 - (1-\alpha)(a_i - y_i)^2$$

Solving the game

Let us keep our assumption that $w_{ij} = 1/d_i$ for all $g_{ij} = 1$.

- We can rewrite our system of best responses in matrix form.

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- $(\mathbf{I} - \alpha \mathbf{W})\mathbf{a} = \mathbf{b}$
- We need to invert the matrix $(\mathbf{I} - \alpha \mathbf{W})$.

Solving the game - contd

- All rows of $\mathbf{W}$ sum up to 1 (row stochastic matrix).
- $\alpha \in (0, 1)$.

Result

The game has a unique Nash equilibrium given by:

$$\mathbf{a}^* = (\mathbf{I} - \alpha\mathbf{W})^{-1}\mathbf{b}$$

Moreover, $(\mathbf{I} - \alpha\mathbf{W})$ is a non-negative matrix.

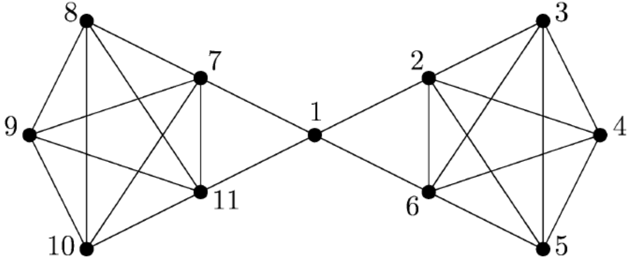
Further result

Nash equilibrium and network structure

The actions in equilibrium are directly proportional to the Bonacich centrality.

- This model has first been studied by *Ballester, Calvo-armengol and Zenou, ECMA, 2006*.

The key player

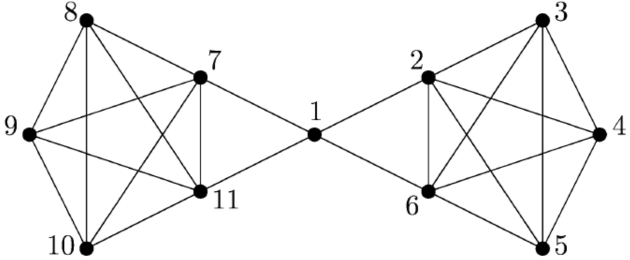


Who is the key player?

- Player 1 connects the network
- Removing player 2 we disrupt the highest number of links

Who has the highest impact on direct and indirect effects?

The key player



Who is the key player? Result from BCZ 2006

- If α is low enough player 2 is the key player
- If α is large enough, player 1 is the key player $\Longrightarrow$ Indirect effects matter more

Bramoullè, Kranton, D'Amours 2014

Standard notation:

- n agents
- Agents choose an action $x_i \geq 0$, $\mathbf{x}_{-i}$ denotes the vector of actions other than i.
- $g_{ij} = g_{ji} = 1$ indicates a link between i and j. Undirected network.
- $\delta > 0$ measure how much i and j affect each other's payoff.

Payoff

Focus on linear best reply games.

A public good game

$$U_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = b_i(x_i + \delta \sum_j g_{ij} x_j) - \kappa_i x_i$$

where κ_i is the marginal cost.

Best replies

$$f_i(\mathbf{x}_{-i}, \delta, \mathbf{G}) = \bar{x}_i - \delta \sum_j g_{ij} x_j \quad \text{if} \quad \delta \sum_j g_{ij} x_j < \bar{x}_i$$

and $f_i = 0$ otherwise.

$\bar{x}_i$ is the value equating marginal cost (d) and marginal benefit (a), assumed to be equal among players.

Payoff - II

- We can obtain the same best reply function by considering the following payoff:

$$\tilde{U}_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = \bar{x}_i x_i - \frac{1}{2}x_i^2 - \delta \sum_j g_{ij} x_i x_j$$

- More models can be considered (e.g. Cournot model with substitutes)

Nash equilibria

Two definitions:

- $\mathbf{G}_A$ denotes the links connecting active agents.
- $\mathbf{G}_{N-A,A}$ denotes the links connecting inactive agents to active agents.

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Result

Actions $\mathbf{x}$ with active agents A is a Nash equilibrium if and only if

(i) $(\mathbf{I} + \delta \mathbf{G}_A)\mathbf{x}_A = \mathbf{1}$

(ii) $\delta \mathbf{G}_{N-A,A}\mathbf{x}_A \geq \mathbf{1}$

This generalize previous results, for which δ was either high (*Bramoulle Kranton, 2007*) or low (*Ballester, Calvo-Armengol, Zenou, 2006*).

Examples

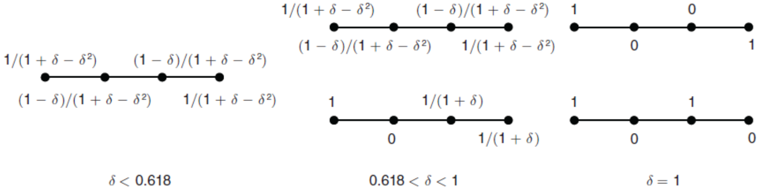

Figure 1. Equilibria in a Line Network

Examples

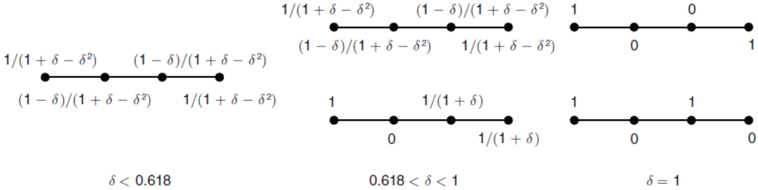

FIGURE 1. EQUILIBRIA IN A LINE NETWORK

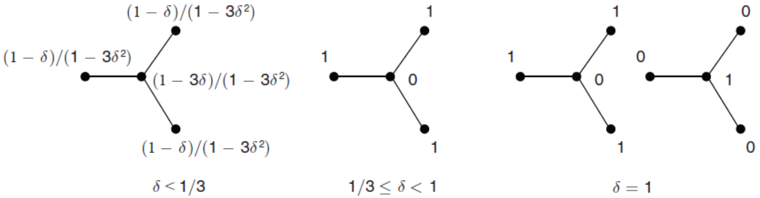

FIGURE 2. EQUILIBRIA IN A STAR NETWORK

Overview - Potential function

For further results, we need to define conditions for getting all Nash equilibria.

Potential games (Monderer, Shapley, 1996)

Consider a game with payoffs $V_i(x_i, \mathbf{x}_{-i})$. A function $\varphi(x_i, \mathbf{x}_{-i})$ is a *potential function* if and only if, for all x_i, x_i' and all $\mathbf{x}_{-i}$:

$$\varphi(x_i, \mathbf{x}_{-i}) - \varphi(x_i', \mathbf{x}_{-i}) = V_i(x_i, \mathbf{x}_{-i}) - V_i(x_i', \mathbf{x}_{-i}), \forall i$$

It exists if and only if

$$\partial^2 V_i(\mathbf{x})/\partial x_i \partial x_j = \partial^2 V_j(\mathbf{x})/\partial x_j \partial x_i$$

The potential function mimics individual best responses, and therefore allows for simplified analysis. We can find equilibrium actions, given all vectors of other agents' actions.

Results - Potential function

Take the game with quadratic payoffs $\tilde{U}_i$. It has a *potential function* since $\partial^2 \tilde{U}_i(\mathbf{x})/\partial x_i \partial x_j = \partial^2 \tilde{U}(\mathbf{x})/\partial x_j \partial x_i = -\delta g_{ij}$, and it is

$$\varphi(\mathbf{x}; \delta, \mathbf{G}) = \mathbf{x}^T \mathbf{1} - \frac{1}{2} \mathbf{x}^T (\mathbf{I} + \delta \mathbf{G}) \mathbf{x}$$

Consider maximizing the *potential function*. We have then the following result:

LEMMA 1

The set of Nash equilibria for a given $\mathbf{G}$ and δ corresponds to the set of maxima and saddle points of the potential function $\varphi(\mathbf{x}; \delta, \mathbf{G})$ on $\mathbb{R}^n_+$.

Results - Uniqueness and lowest eigenvalue

Given *LEMMA 1* a sufficient condition for uniqueness is concavity of the *potential function*. The matrix of second derivatives is given by:

$$\bigtriangledown^2 \varphi = -(\mathbf{I} + \delta \mathbf{G})$$

Therefore $\mathbf{I} + \delta \mathbf{G}$ must be positive definite, which is equivalent to $1 + \delta \lambda_{\min}(\mathbf{G}) > 0$.
Result follows straightforward:

PROPOSITION 2
If $|\lambda_{\min}(\mathbf{G})| < 1/\delta$, there is a unique Nash equilibrium.

Intuition
If cumulative effect of the ups and downs is high enough (lowest eigenvalue reflects substitutability), multiple equilibria emerge. the larger $\lambda_{\min}$, the greater the amplification of actions.

Stability in an example

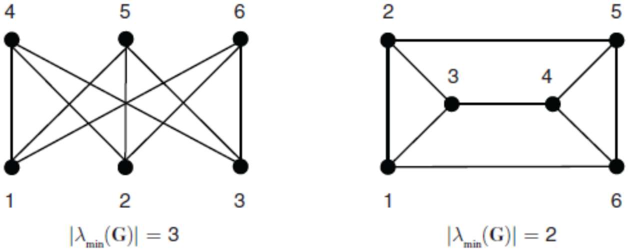

FIGURE 4. LOWEST EIGENVALUE AND STABLE EQUILIBRIA

Imagine a “perturbation” ε in agent’s 1 best reply. What happens?

Results - Stable equilibria

To study stability Nash tâtonnement.
Lowest eigenvalue again is the key of stability.

Stable equilibria and Lowest eigenvalue

An equilibrium **x** with active agents A and inactive agents is stable if and only if $|\lambda_{\mathsf{min}}(\mathbf{G}_A)| < 1/\delta$.
When $|\lambda_{\mathsf{min}}(\mathbf{G})| < 1/\delta$, the unique equilibrium is stable.
When $|\lambda_{\mathsf{min}}(\mathbf{G})| > 1/\delta$, among the multiple equilibria, all stable equilibria involve inactive agents.

Intuition

We check the perturbation ε of an equilibrium that yields the largest change in best replies (*maximal perturbation*). It derives from the eigenvectors associated to the lowest eigenvalue (because Str. Sub.). If the maximal perturbation does not lead the system away, nothing else will.

Results - Lowest eigenvalue and network structure

Take the characterization of the lowest eigenvalue:

$$\lambda_{\min}(\mathbf{G}) = \min \boldsymbol{\varepsilon}^T \mathbf{G} \boldsymbol{\varepsilon}$$

Now divide agents in two sets (bipartite graph) R and S such that $R = \{i | \varepsilon_i \geq 0\}$ and $S = \{i | \varepsilon_i < 0\}$. Then

$$\lambda_{\min}(\mathbf{G}) = \sum_{i,j \in R} \varepsilon_i \varepsilon_j g_{ij} + \sum_{i,j \in S} \varepsilon_i \varepsilon_j g_{ij} + 2 \sum_{i \in R, j \in S} \varepsilon_i \varepsilon_j g_{ij}$$

Intuition (PROP. 7)

First two terms captures the link within, and they are always positive. The third term captures between links, and when is larger in magnitude lowers the value of $\lambda_{\min}(\mathbf{G})$. This means that actions rebound more in opposite directions (**substitutability**), and it happens when more links between groups are present.

What's next?

Current research has some open questions:

- Integrate behavior with network formation
- Take models of network formation to the data: structural modeling of peer effects
- Study the impact of homophily, clustering and other network characteristics on behavior

Thank you for your attention